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Inconsistency of a proposed constrained dynamics for the damped harmonic oscillator

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Abstract. It is argued that the method introduced recently by Wang to quantise the damped harmonic oscillator is untenable because it does not reproduce the standard results for the quantum oscillator in the limit of arbitrarily small damping. The method is also shown to be inconsistent if the friction coefficient is allowed to take any positive value. Furthermore, it is established that no moving wavepacket can be constructed within Wang's formalism.

1. Introduction

The phenomenological quantisation of dissipative systems is still being investigated from several different points of view. The canonical quantisation based on Bateman's time-dependent Lagrangian (Bateman 1931, Kanai 1948) has been criticised as actually referring to a variable-mass system (Greenberger 1979a, Ray 1979). Other suggestions then followed to attack the problem such as taking the mass as a dynamical variable (Greenberger 1979b), or introducing nonlinear Schrödinger equations (Kostin 1972, Hasse 1975, Skagerstam 1977, Stocker and Albrecht 1979, Schuch *et al* 1983, 1984, Brül and Lange 1984). The inclusion of an external stochastic force has been studied, chiefly to overcome certain difficulties engendered by Bateman's Lagrangian (Svin'in 1976, Messer 1979), while the exploitation of the quantum Liouville equation has also been tried (Brinati and Mizrahi 1980). Since the variety of approaches is quite broad, the reader is referred to Dekker's review (Dekker 1981), where an extensive survey of the literature can be found. As to the phenomenological canonical quantisation of dissipative (more generally, non-conservative) systems, however, it has been repeatedly shown to be impossible or ambiguous (Brittin 1950, Havas 1956, Messer 1978, Edwards 1979, Lemos 1981a, b). The results of Brittin (1950) are partially corrected in Lemos (1981a).

The great importance of constrained Hamiltonian systems is witnessed by their occurrence in the modern attempts to describe the fundamental interactions of elementary particles. The canonical formulation of gauge and string theories, for instance, leads naturally to constraints, and Dirac's formalism (Dirac 1964, Sundermeyer 1982) to deal with them is usually invoked. Sometimes it is useful to introduce artificial constraints into certain systems with the intention of bringing forth new symmetries that may simplify the analysis of the dynamics of such systems. This is customarily done at the cost of an enlargement of the phase space of the system through additional degrees of freedom that are eventually eliminated. Recently, a new treatment (Wang 1987) of the damped harmonic oscillator has been put forward, regarding it to be a constrained system described by a constrained generalised Hamiltonian. To our knowledge, this is the first attempt so far to construct a constrained Hamiltonian model for the description of a dissipative system at the quantum level. Instead of artificially enlarging the phase space, Wang introduces constraints in the ordinary phase space in such a way that the state vector has to satisfy a nonlinear subsidiary condition, in addition to a linear Schrödinger equation.

What we undertake to show in the present paper is that Wang's theory is untenable on physical grounds, particularly because it does not reproduce the standard results for the quantum oscillator in the limit of vanishing friction coefficient. We also adduce some general arguments to the effect that any theory constructed along the lines suggested by Wang is bound to fail. Finally, in the appendix it is pointed out that no moving wavepacket can exist in Wang's theory, and this constitutes a further strong reason against its viability.

2. A proposed constrained dynamics for the quantised damped harmonic oscillator

In a recent paper, Wang (1987) proposed a new and ingenious method to quantise the damped harmonic oscillator by considering the equation of motion

$$\ddot{x}+\gamma\dot{x}+\omega^2x=0 \tag{1}$$

as arising from a constrained Hamiltonian system in the bidimensional phase space (x, p). His classical treatment introduces a first-class constraint ϕ_1 in phase space. According to Dirac's theory of constrained systems, in the quantised theory the first-class constraints must be imposed as supplementary conditions on the physical state belonging to the Hilbert space $\mathscr{H}=L^2(\mathbb{R})$. Let $\hat{H}_0$ be the Hamiltonian operator of a harmonic oscillator of frequency ω, that is,

$$\hat{H}_0=\frac{\hat{p}^2}{2m}+\frac{m\omega^2}{2}\hat{x}^2 \tag{2}$$

where $\hat{x}$ and $\hat{p}$ are self-adjoint operators obeying the usual canonical commutation relations. Then Wang's supplementary condition (3.17) on the wavefunction becomes

$$\hat{\phi}_1\psi=\hat{H}_0\psi-\mathrm{i}\hbar\gamma\omega[\ln\psi-\tfrac{1}{2}\ln(\psi^*\psi)]=0. \tag{3}$$

In addition, ψ obeys a linear Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t}=\hat{H}\psi \tag{4}$$

where the explicit form of the Hamiltonian operator $\hat{H}$ will be irrelevant for our purposes.

To our knowledge, all phenomenological quantum models of the damped harmonic oscillator enjoy the natural property that as $\gamma\to 0$ any state vector ψ_γ reduces to a state vector ψ_0 of the undamped oscillator. This is assumed by Wang himself in so far as he considers the case of small damping, with $\delta=\gamma/(\omega^2-\gamma^2/4)^{1/2}\ll 1$, and makes use of a perturbative approximation method by expanding the Hamiltonian operator and the wavefunction in powers of the small parameter δ. To the first order in δ one should have [see Wang's equation (3.5)]

$$\psi=\psi_0+\delta\psi_1 \qquad \psi_0,\psi_1\in\mathscr{H} \tag{5}$$

where ψ_0 obeys the Schrödinger equation

$$i\hbar\frac{\partial\psi_0}{\partial t}=\hat{H}_0\psi_0. \tag{6}$$

In other words, ψ_0 is a state vector of the undamped oscillator.

Therefore, not being aware of any convincing argument to the contrary, we shall adopt the natural point of view that any admissible quantum theory of the damped harmonic oscillator must reproduce the ordinary theory of the quantum oscillator as the friction coefficient γ tends to zero. What we intend to prove in the sequel is that the quantisation scheme devised by Wang does not meet this requirement, and for this reason must be dismissed as unacceptable. As a byproduct of our investigation, it will also be shown that Wang's theory may only be consistent for a restricted set of values of the friction coefficient γ, and this is a further fatal objection against it.

3. Inconsistency of the theory

Let us define the nonlinear operator $\hat{\mathscr{O}}$ by

$$\hat{\mathscr{O}}\psi=\psi[\ln\psi-\tfrac{1}{2}\ln(\psi^*\psi)]. \tag{7}$$

In order to ascribe a precise meaning to $\hat{\mathscr{O}}$, and therefore to equation (3), we shall take the principal determination of the logarithm (Markushevich 1970), as this seems to be the most natural choice. Thus, with

$$\psi=|\psi|\,e^{i\theta} \qquad -\pi<\theta\leqslant\pi \tag{8}$$

we have

$$\hat{\mathscr{O}}\psi=i\theta\psi. \tag{9}$$

With such a definition the operator $\hat{\mathscr{O}}$ has a few interesting properties, which we now explore. From equations (8) and (9) it follows that

$$|(\hat{\mathscr{O}}\psi)(x)|=|\theta(x)|\,|\psi(x)|\leqslant\pi|\psi(x)| \tag{10}$$

whence

$$\|\hat{\mathscr{O}}\psi\|\leqslant\pi\|\psi\| \tag{11}$$

showing that $\hat{\mathscr{O}}$ is a bounded operator. This does not mean that $\hat{\mathscr{O}}$ is necessarily continuous, because it is not a linear operator. Notice further that

$$\hat{\mathscr{O}}(\lambda\psi)=\lambda\hat{\mathscr{O}}\psi \qquad \lambda>0 \tag{12}$$

so that $\hat{\mathscr{O}}$ is a positive-homogeneous operator, in spite of its nonlinear character. By rewriting equation (3) in the form

$$\hat{H}_0\psi-i\hbar\gamma\hat{\mathscr{O}}\psi=0 \tag{13}$$

the linearity of $\hat{H}_0$, together with the positive homogeneity of $\hat{\mathscr{O}}$, allows us to require that any solution $\psi_\gamma\in\mathscr{H}$ to equation (13) be chosen in such a way that $\|\psi_\gamma\|=1$. Since the Schrödinger equation (4) conserves probability, we may always regard as normalised any simultaneous solution to equations (3) and (4).

Before going forward to what we want to establish, it is necessary to consider an ancillary result.

Lemma. Let ψ be any normalised vector of the Hilbert space of states $\mathscr{H}$. Then

$$\|\hat{H}_0\psi\|^2 \geqslant \tfrac{1}{4}\hbar^2\omega^2. \tag{14}$$

Proof. Let $\{\varphi_n\}_{n=0}^{\infty}$ be the orthonormal basis of $\mathscr{H}$ made up with the eigenvectors φ_n of $\hat{H}_0$, which are such that

$$\hat{H}_0\varphi_n = (n+\tfrac{1}{2})\hbar\omega\varphi_n. \tag{15}$$

It is possible to write

$$\psi = \sum_{n=0}^{\infty} C_n\varphi_n \qquad C_n \in \mathbb{C} \tag{16}$$

with

$$\|\psi\|^2 = \sum_{n=0}^{\infty} |C_n|^2 = 1. \tag{17}$$

Therefore,

$$\hat{H}_0\psi = \sum_{n=0}^{\infty} C_n(n+\tfrac{1}{2})\hbar\omega\varphi_n \tag{18}$$

so that

$$(\hat{H}_0\psi, \hat{H}_0\psi) = \sum_{n=0}^{\infty} \hbar^2\omega^2(n+\tfrac{1}{2})^2|C_n|^2 \geqslant \frac{\hbar^2\omega^2}{4}\sum_{n=0}^{\infty}|C_n|^2 = \tfrac{1}{4}\hbar^2\omega^2 \tag{19}$$

and the proof is complete.

We are now prepared to state and prove our main results.

Theorem. Let $\psi_0 \in \mathscr{H}$ be a normalised state vector of the usual harmonic oscillator. Then there exists no solution $\psi_\gamma \in \mathscr{H}$ to Wang's equations (4) and (13) such that $\psi_\gamma \to \psi_0$ as $\gamma \to 0$.

Proof. As we have previously remarked, any solution $\psi_\gamma \in \mathscr{H}$ to equations (4) and (13) can be taken to be normalised. Accordingly, let us assume that $\|\psi_\gamma\| = 1$ and rewrite equation (13) in the form

$$\hat{\mathcal{O}}\psi_\gamma = \frac{1}{\mathrm{i}\hbar\gamma}\hat{H}_0\psi_\gamma. \tag{20}$$

This equation combined with equation (11) leads to

$$\|\psi_\gamma\|^2 \geqslant \frac{1}{\pi^2}\|\hat{\mathcal{O}}\psi_\gamma\|^2 = \frac{1}{\pi^2\hbar^2\gamma^2}\|\hat{H}_0\psi_\gamma\|^2. \tag{21}$$

Having recourse to the lemma we conclude that

$$\|\psi_\gamma\|^2 \geqslant \frac{\omega^2}{4\pi^2\gamma^2} \xrightarrow[\gamma\to 0]{} \infty. \tag{22}$$

This contradicts the assumption that ψ_γ is normalised and, moreover, shows that ψ_γ does not converge to an element of $\mathscr{H}$ as $\gamma \to 0$. The proof is complete.

Corollary. There exists no non-trivial solution to equation (13) in $\mathscr{H}$ if $\gamma < \omega/2\pi$.

Proof. Let $\psi_\gamma \in \mathscr{H}$ be a non-trivial solution (not necessarily normalised) to equation (13). From equations (21), (19) and (17) it follows at once that

$$\|\psi_\gamma\|^2 \geqslant \frac{\omega^2}{4\pi^2\gamma^2}\|\psi_\gamma\|^2 \tag{23}$$

hence

$$\gamma \geqslant \omega/2\pi. \tag{24}$$

This restriction on the allowed values of γ implies the rejection of Wang's theory as unphysical, and characterises as meaningless the perturbative approximation scheme employed in his paper, since it presupposes the validity of the model for γ/ω arbitrarily small.

The following remarks concerning the above results are in order. The theorem retains its validity if some other branch of the logarithm is chosen, whereas the bound on γ expressed in the corollary is lowered. It is clear that our method of proof allows us to infer a lower positive bound on γ only when a particular choice of branch of $\ln z$ is made. Even if different branches of the logarithmic function are selected for various values of x to make the phase $\theta(x)$ continuous, the only reasonable limit of the constraint equation (3) seems to be $\hat{H}_0\psi = 0$, which entails $\psi = 0$. This reasoning, that, of course, is not meant to be rigorous, suggests that quite generally the constraint introduced by Wang prevents his theory from having an acceptable limit as the friction coefficient tends to zero.

In the appendix it is further shown in full generality that the constraint equation (3) is not compatible with the existence of moving wavepackets. We believe that this is another strong reason to regard Wang's theory as physically unacceptable.

4. Conclusion

For one-dimensional non-conservative systems it is not difficult to understand why any attempt along the lines suggested by Wang will inevitably fail. If one insists that one is dealing with a genuine Hamiltonian system, although constrained, it must be possible to solve the constraint equations and go over to a reduced phase space (x^*, p^*) endowed with a Hamiltonian $H^*(x^*, p^*)$ and where there are no constraints. Making use of the path-integral quantisation method, for instance, the formula for the propagator in the reduced phase space can be expressed in terms of the original phase space at the expense of a modification of the integration measure (Faddeev 1969). In any case, if the original phase space has dimension $2N$, the reduced one has at most dimension $2N-2$, if there is only one first-class constraint. The situation considered by Wang corresponds to $N=1$, and in the most favourable circumstances one would end up with a zero-dimensional phase space. Of course, even at the classical level such a theory is devoid of physical content.

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Appendix: The non-existence of moving wavepackets

By making use of equations (2) and (9) the constraint equation (3) can be written in the form

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{m\omega^2}{2}x^2\psi+\hbar\gamma\theta\psi=0. \tag{A1}$$

Let us put $\psi = U+\mathrm{i}V$, where U and V are real functions. By taking real and imaginary parts of equation (A1) we readily obtain the following equations:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 U}{\partial x^2}+\frac{m\omega^2x^2}{2}U+\hbar\gamma\theta U=0 \tag{A2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 V}{\partial x^2}+\frac{m\omega^2x^2}{2}V+\hbar\gamma\theta V=0. \tag{A3}$$

We now take the product of equation (A2) by V, of equation (A3) by U and subtract the resulting equations, obtaining

$$V\frac{\partial^2 U}{\partial x^2}-U\frac{\partial^2 V}{\partial x^2}=0. \tag{A4}$$

This last result is equivalent to

$$\frac{\partial}{\partial x}\left(V\frac{\partial U}{\partial x}-U\frac{\partial V}{\partial x}\right)=0 \tag{A5}$$

whence

$$V\frac{\partial U}{\partial x}-U\frac{\partial V}{\partial x}=\alpha(t) \tag{A6}$$

where $\alpha(t)$ is an arbitrary function. If ψ is to be square integrable, both functions U and V (and their derivatives) must vanish as $|x|\to\infty$. Then we conclude that $\alpha=0$ and are led to

$$V\frac{\partial U}{\partial x}=U\frac{\partial V}{\partial x} \tag{A7}$$

whose solution is immediate:

$$V=\beta(t)U \tag{A8}$$

where $\beta(t)$ is an arbitrary real function.

From equation (A8) it follows that the wavefunction ψ takes the form

$$\psi=\xi(t)U \qquad \xi(t)=1+\mathrm{i}\beta(t). \tag{A9}$$

As a consequence, the mean value of the linear momentum is given by

$$\begin{aligned}\langle p\rangle_t &= \int_{-\infty}^{\infty}\psi^*\left(-\mathrm{i}\hbar\frac{\partial\psi}{\partial x}\right)\mathrm{d}x=-\mathrm{i}\hbar|\xi(t)|^2\int_{-\infty}^{\infty}U\frac{\partial U}{\partial x}\mathrm{d}x\\ &= -\mathrm{i}\hbar|\xi(t)|^2[\tfrac{1}{2}U^2]_{x=-\infty}^{x=\infty}=0\end{aligned} \tag{A10}$$

by virtue of the boundary conditions satisfied by U at $x=\pm\infty$. Therefore, any conceivable wavepacket in Wang's theory has vanishing expectation value of its linear momentum or, in other words, moving wavepackets are forbidden.

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